Second-Order Uniaxial Perfectly Matched Layer for Finite-Element Time-Domain Methods

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There has always been a compromise between the regular and complex frequency shifted (CFS) perfectly matched layers (PMLs) in simulation of open-region problems in electromagnetics. They fail to provide accurate results when evanescent and low-frequency propagating waves exist in the problem. A PML with metrics composed of multiple poles, i.e., high-order PML, has shown great performance in such situations. In this paper, we extend the application of the second-order PML to both mixed $E-B$ finite-element time-domain (FETD) and vector wave equation (VWE) FETD methods. In order to have an efficient and easier-to-implement formulation, the Möbius transformation technique is employed in discretization of the PML metrics. A numerical result is provided to demonstrate the validity of the formulation.

***Index Terms*—Anisotropic media, mixed finite-element time-domain (FETD), high-order perfectly matched layer (PML).**

I. Introduction

BÉRENGER introduced perfectly matched layer (PML) as a powerful technique for the finite-difference time-domain (FDTD) simulation of open-region problems in electromagnetics [1]. The original PML is capable of absorbing propagating waves of any frequency; however, it does not show a good performance in case of evanescent waves. Various attempts have been made to alleviate this drawback among which the complex frequency shifted (CFS)-PML is the most widely-accepted approach [2]. Despite the great performance of the CFS-PML for evanescent waves, it has a poor absorption for low-frequency propagating waves. Hence, in a general problem where both evanescent and low-frequency waves exist, both regular and CFS-PML can not provide an accurate solution.

In order to retain the advantages of both PMLs, the idea of using high-order metrics is introduced [3]. It is shown that employing a second-order metric can effectively absorb both low-frequency and evanescent waves at the same time [3], [4]. Unfortunately, the majority of these formulations have been proposed for the FDTD. In [5], the authors extended second-order PML to the finite-element time-domain (FETD) based on the vector wave equation (VWE) using the recursive convolution technique, which results in very long and complicated terms.

This paper aims at developing a second-order PML for the mixed FETD formulation for the first time. The Möbius transformation technique is utilized to discretize the PML metrics, which allows a systematic and efficient implementation. Moreover, the idea is also extended to the VWE-FETD formulation, which is simpler than the approach proposed in [5].

II. Formulation

Discretization of the PML metric and underlying FETD formulations are presented in this section.

A. PML Metrics

For a second-order PML, the metric can be generally stated as

$$\gamma_x(\omega) = \left(\kappa_1 + \frac{\sigma_1}{\alpha_1 + j\omega\epsilon}\right)\left(\kappa_2 + \frac{\sigma_2}{\alpha_2 + j\omega\epsilon}\right) \tag{1}$$

where $\kappa_{1,2}$, $\alpha_{1,2}$, and $\sigma_{1,2}$ control the performance of the PML. Particularly, α specifies the frequency shift in the CFS-PML and setting it to zero yields the regular PML metric.

Uniaxial PML (UPML) can be considered as an anisotropic dispersive medium with the following material properties: $\overline{\overline{\varepsilon}} = \overline{\overline{\varepsilon}}_b\overline{\overline{\Lambda}}$ and $\overline{\overline{\mu}} = \overline{\overline{\mu}}_b\overline{\overline{\Lambda}}$, in which $\overline{\overline{\varepsilon}}_b$ and $\overline{\overline{\mu}}_b$ denote background material permittivity and permeability. $\overline{\overline{\Lambda}}$ represents the PML tensor, as

$$\overline{\overline{\Lambda}} = \begin{bmatrix} \dfrac{\gamma_y\gamma_z}{\gamma_x} & 0 & 0 \\ 0 & \dfrac{\gamma_x\gamma_z}{\gamma_y} & 0 \\ 0 & 0 & \dfrac{\gamma_x\gamma_y}{\gamma_z} \end{bmatrix} \tag{2}$$

in order to discretize the permittivity and permeability efficiently, we employ the Möbius transformation technique in which we simply replace $j\omega$ terms by

$$\frac{2}{\Delta t}\frac{1-z^{-1}}{1+z^{-1}} \tag{3}$$

which yields a rational function along each direction, $k = x, y, z$, in the z-domain, as

$$\varepsilon_k(z) = \frac{c_{0_k} + c_{1_k}z^{-1} + \cdots + c_{(p_k)_k}z^{-p_k}}{1 + d_{1_k}z^{-1} + \cdots + d_{(p_k)_k}z^{-p_k}} \tag{4}$$

$$\mu_k^{-1}(z) = \frac{q_{0_k} + q_{1_k}z^{-1} + \cdots + q_{(p_k)_k}z^{-p_k}}{1 + r_{1_k}z^{-1} + \cdots + r_{(p_k)_k}z^{-p_k}} \tag{5}$$

B. FETD Formulations

In this section, we briefly explain the formulation of the PML in the mixed FETD. The VWE-FETD formulation will be discussed in the long version of the paper.

In the mixed FETD, the Maxwell equations can be written separately from the constitutive relations as

$$\frac{\partial\{d\}}{\partial t} = [\mathcal{C}]^T\{h\}, \frac{\partial\{b\}}{\partial t} = -[\mathcal{C}]\{e\} \tag{6}$$

where $[\mathcal{C}]$ represents the discrete curl operator. The constitutive relations along each direction, k, are

$$\{d_k\} = \varepsilon_k[\mathcal{M}_k]\{e\}, \{h_k\} = \mu_k^{-1}[\mathcal{M}_{f_k}]\{b\} \tag{7}$$

where

$$\mathcal{M}_k^{i,j} = \int_\Omega \left(\boldsymbol{W}_i^{(1)} \cdot \hat{u}_k\right)\left(\hat{u}_k \cdot \boldsymbol{W}_j^{(1)}\right) dV \tag{8}$$

$$\mathcal{M}_{f_k}^{i,j} = \int_\Omega \left(\boldsymbol{W}_i^{(2)} \cdot \hat{u}_k\right)\left(\hat{u}_k \cdot \boldsymbol{W}_j^{(2)}\right) dV \tag{9}$$

In order to implement (7) in an efficient manner, we use a memory-efficient approach known as the transposed direct form II [6], which yields

$$\{d_k\}^{n+1} = c_{0k}[\mathcal{M}_k]\{e\}^{n+1} + \{\mathcal{W}_{1,k}\}^n \tag{10a}$$
$$\{h_k\}^{n+1} = q_{0k}[\mathcal{M}_{f_k}]\{b\}^{n+1} + \{\mathcal{G}_{1,k}\}^n \tag{10b}$$

where the auxiliary variables $\{\mathcal{W}\}$ and $\{\mathcal{G}\}$ are updated as

$$\begin{aligned} \{\mathcal{W}_{\alpha,k}\}^n &= c_{\alpha_k}[\mathcal{M}_k]\{e\}^n - d_{\alpha_k}\{d_k\}^n + \{\mathcal{W}_{\alpha+1,k}\}^{n-1} \\ &\qquad ;\alpha = 1, 2, \cdots, p_k - 1 \\ \{\mathcal{W}_{\alpha,k}\}^n &= c_{\alpha_k}[\mathcal{M}_k]\{e\}^n - d_{\alpha_k}\{d_k\}^n \qquad ;\alpha = p_k \end{aligned} \tag{11}$$

$$\begin{aligned} \{\mathcal{G}_{\alpha,k}\}^n &= q_{\alpha_k}[\mathcal{M}_{f_k}]\{b\}^n - r_{\alpha_k}\{h_k\}^n + \{\mathcal{G}_{\alpha+1,k}\}^{n-1} \\ &\qquad ;\alpha = 1, 2, \cdots, p_k - 1 \\ \{\mathcal{G}_{\alpha,k}\}^n &= q_{\alpha_k}[\mathcal{M}_{f_k}]\{b\}^n - r_{\alpha_k}\{h_k\}^n \qquad ;\alpha = p_k \end{aligned} \tag{12}$$

Having substituted (11) and (12) in (6) and discretizing it using the leap-frog scheme, or another method like the Crank-Nicolson (CN)-FETD [7], the main update equation is obtained as

$$\begin{aligned} [\mathcal{M}_t]\{e\}^{n+1} &= [\mathcal{M}_t]\{e\}^n + \Delta t[\mathcal{C}]^T[\mathcal{M}_{f_t}]\{b\}^{n+1/2} \\ &\quad - \{\mathcal{W}_t\}^n + \{\mathcal{W}_t\}^{n-1} + \Delta t[\mathcal{C}]^T\{\mathcal{G}_t\}^{n-1/2} \end{aligned} \tag{13}$$

$$\{b\}^{n+3/2} = \{b\}^{n+1/2} - \Delta t[\mathcal{C}]\{e\}^{n+1} \tag{14}$$

where

$$[\mathcal{M}_{f_t}] = \sum_k q_{0_k}[\mathcal{M}_{f_k}], \quad [\mathcal{M}_t] = \sum_k c_{0_k}[\mathcal{M}_k] \tag{15a}$$

$$\{\mathcal{W}_t\}^n = \sum_k \{\mathcal{W}_{1,k}\}^n, \quad \{\mathcal{G}_t\}^n = \sum_k \{\mathcal{G}_{1,k}\}^n \tag{15b}$$

Once $\{e\}$ abd $\{b\}$ are updated using (13) and (14), they are used to update the auxiliary variables $\{\mathcal{W}\}$ and $\{\mathcal{G}\}$ required for the next time step.

III. Numerical Result

The numerical example involves a 3-D parallel-plate waveguide whose sides are both terminated with second-order PML walls. The waveguide is excited with several dipoles in different directions in the middle of the waveguide to stimulate different modes. The spectrum of the pulse is adjusted such that it covers both evanescent and propagating modes. The reference solution is obtained by extending the waveguide from both sides. The relative reflection error (in dB) is calculated using

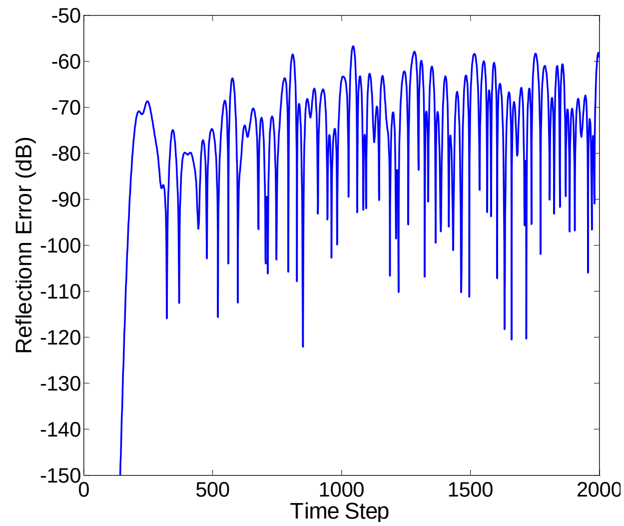

Fig. 1. Relative reflection error.

$$R_{dB}(t) = 20log_{10}\left(\frac{|E_{FETD}(t) - E_{ref}(t)|}{max\left(|E_{ref}(t)|\right)}\right). \tag{16}$$

Fig. 1 shows the relative reflection error within 2000 time steps. It is less than -50 dB, which shows a satisfactory performance.

IV. Conclusion

The second-order UPML has been extended to the mixed FETD method in an efficient manner using the Möbius transformation technique. A similar approach has been also utilized for the FETD method based on the VWE. Although the second-order PML is considered here, it can be readily extended to higher-order PMLs as well.

Complimentary formulations will be presented and a study between the performance of regular, CFS, and second-order PMLs will be conducted at the conference and in the long version of the paper.

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